METHOD OF GROUP DIFFERENCES FOR THE CONSTRUCTION OF THREE ASSOCIATE CYCLIC-DESIGNS

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Introduction

The cyclic designs were originally obtained by Nandi & Adhikary (1970) and Adhikary (1966) in systematic manner. However, various particular cases of these designs have been previously obtained by many authors [Roy (1955), Raghavaro (1960), Singh and Singh (1965) etc.].

The method of differences for the construction of incomplete block designs is by now a standard method. A complete bibliography is available in Hall, M. Jr. (1967). Bruck (1955) extended the idea of method of differences to that of group differences. Here we shall consider different methods of group differences for the construction of cyclic designs of various types. Proof of the general theorems stated here will follow the usual lines laid down for such results on the method of differences, and as such they will be omitted. Each method has been explained by means of an example.

2.1 *Cyclic association scheme* (*First type*) :

2.1.1. Consider an abelian group G of order $v=m_1n$. Let it be possible to decompose G into direct factors : $G=G_1 \otimes G_2$ where G_1 consists of m_1 elements $1, d_1, d_2, \ldots\ldots, d_{m_1}-1$, while the non-unit elements of G_2 can be divided into two disjoint sets A and B of m_2 and m_3 elements respectively, i.e., $A=(e_1, e_2, \ldots, e_{m_2})$ and $B=(f_1, f_2, \ldots, f_{m_3})$. Further, let the elements of A be such that all the inverse elements are also in A and that among the m_2 (m_2-1) non-unit ratios arising out of them, the elements of A are repeated α times and those of B β times each. Obviously, $n=m_2+m_3+1$, $m_2\alpha+m_3\beta=m_2(m_2-1)$.

With each element of G let us associate one treatment.

Let the first associates of any treatment θ be $\theta\,(G_1-1)$, its second associates be $\theta\, G_1 \otimes A$ and its third associates $\theta\, G_1 \otimes B$.

The parameters of the association scheme are

$$v=m_1.n,\; n_1=m_1-1,\; n_2=m_1 m_2,\; n_3=m_1 m_3.$$

$$p^1_{jk}=\begin{bmatrix} m_1-2 & 0 & 0 \\ 0 & m_1m_2 & 0 \\ 0 & 0 & m_1m_3 \end{bmatrix}$$

$$p^2_{jk}=\begin{bmatrix} 0 & m_1-1 & 0 \\ m_1-1 & m_1\alpha & m_1(m_2-\alpha-1) \\ 0 & m_1(m_2-\alpha-1) & m_1(m_3-m_2+\alpha+1) \end{bmatrix}$$

$$p^3_{jk}=\begin{bmatrix} 0 & 0 & m_1-1 \\ 0 & m_1\beta & m_1(m_2-\beta) \\ m_1-1 & m_1(m_2-\beta) & m_1(m_3-m_2+\beta-1 \end{bmatrix}$$

Theorem 2.1.1. Let it be possible to find a set of t blocks B_1, B_2, .. , Bt satisfying the following conditions :

(*i*) Every block contains k treatments (the treatments contained in the same block being different from one another).

(*ii*) Among the $tk\,(k-1)$ ratios arising out of these t-blocks, the non-unit elements of G_1 appear λ_1-times the elements of $G_1 \otimes A$ appear λ_2 times and the elements of $G_1 \otimes B$ appear λ_3 times.

Then on developing these blocks (i.e. multiplying successively by the elements of the group) we get the solution of the design whose parameters of the second kind are :

$$v=m_1n,\; b=vt,\; r=kt,\; k,\; \lambda_1,\; \lambda_2,\; \lambda_3.$$

Example 1. Consider the group G formed by the elements a, c and their different powers when

$$a^{13}=c^2=1.$$

Let G_1: 1, c; $A : a^2, a^5, a^6, a^7, a^8, a^{11}$

$$B : a, a^3, a^4, a^9, a^{10}, a^{12}$$

So, $\alpha=2$, $\beta=3$.

Hence the parameters of the association scheme are

$$v=26,\ n_1=1,\ n_2=n_3=12.$$

$$p^1_{jk}=\begin{bmatrix}0 & 0 & 0\\ 0 & 12 & 0\\ 0 & 0 & 12\end{bmatrix};\ p^2_{jk}=\begin{bmatrix}0 & 1 & 0\\ 1 & 4 & 6\\ 0 & 6 & 6\end{bmatrix};\ p^3_{jk}=\begin{bmatrix}0 & 0 & 1\\ 0 & 6 & 6\\ 1 & 6 & 4\end{bmatrix}$$

Consider the initial blocks $(1, a^2, a^2c)$, $(1, a^5, a^5c)$ and $(1, a^6, a^6c)$. On developing these initial blocks we get the solution of the design

$$v=26,\ b=78,\ r=9,\ k=3,\ \lambda_1=6,\ \lambda_2=1,\ \lambda_3=0)$$

2.1.2. Consider the multiplicative abelian group

$$G=[a] \text{ of order } m_2+m_3+1.$$

Let us divide the non-unit elements of G into two disjoint sets A and B with m_2 and m_3 elements respectively so that $A=A^{-1}$ and among the ratios arising out of the elements of A, the elements of A appear α times and the elements of B appear β times.

To any element a^i let there correspond m_1 treatments

$$(a^i)_1,\ (a^i)_2,\ \ldots,\ (a^i)_{m_1}.$$

Let us consider the first associate of

$$(a^i)_j \text{ as } (a^i)_1, (a^i)_2, \ldots, (a^i)_{j-1}, (a^i)_{j+1} \ldots, (a^i)_{m_1},$$

second associate of

$$(a^i)_j \text{ as } (a^i.\ \phi)_j,$$

where

$\phi \in A$ and the rest are third associates.

The parameters of the association scheme are

$$n_1=m_1-1,\ n_2=m_1m_2,\ n_3=m_1m_3$$

$$p^1_{jk}=\begin{bmatrix} m_1-2 & 0 & 0 \\ 0 & m_1m_2 & 0 \\ 0 & 0 & m_1m_3 \end{bmatrix}$$

$$p^2_{jk}=\begin{bmatrix} 0 & m_1-1 & 0 \\ m_1-1 & m_1\,\alpha & m_1(m_2-\alpha-1) \\ 0 & m_1(m_2-\alpha-1) & m_1(m_3-m_2+\alpha+1) \end{bmatrix}$$

$$p^3_{jk}=\begin{bmatrix} 0 & 0 & m_1-1 \\ 0 & m_1\beta & m_1(m_2-\beta) \\ m_1-1 & m_1(m_2-\beta) & m_1(m_3-m_2+\beta-1) \end{bmatrix}$$

Theorem 2.1.2. Let it be possible to select a set of t-blocks $B_1, B_2, \ldots, B_t$ such that

(*i*) Every block contains exactly k treatments (the treatments contained in the same block being different from one another).

(*ii*) Among the kt treatments occurring in the t-blocks exactly r treatments should belong to each of m_1 distinct classes. Obviously, $m_1\, r=kt$.

(Two treatments having the same lower suffix j may be said to belong to the j-th class].

(*iii*) Among the $kt\,(k\text{-}1)$ ratios arising out of these t-blocks, the ratios of the type $l_{i,\,j}$ will appear λ_1 times for $l \in G$ and $i, j=1, 2, \ldots, m_1\ i\neq j$; the ratios of the type a_{ij} where $a\in A$, will appear λ_2 times, for $i, j=1, 2, \ldots, m_1$ and the ratios of the type b_{ij}, when $b \in B$, will appear λ_3 times for $i, j=1, 2, \ldots, m_{\text{j}}$.

So $k\,(k-1)t=m_1(m_1-1)\,\lambda_1+m^2{}_1m_2\lambda_2+m_1{}^2m_3\lambda_3$.

Then on developing these set of t-blocks (i.e. multiplying by elements of G) we get the solution of the design.

$$v=m_1\,(m_2+m_3+1)=m_1n,\ b=nt,\ r,\ k,$$

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ n_1=m_1-1,\ n_2=m_1m_2,\ n_3=m_1m_3,$$

1. By ratio of the type a_{ij} we mean the ratio of two elements x_i and y_j when $x/y=a$.

$$p^1_{jk}=\begin{bmatrix} m_1-2 & 0 & 0 \\ 0 & m_1m_2 & 0 \\ 0 & 0 & m_1m_3 \end{bmatrix}$$

$$p^2_{jk}=\begin{bmatrix} 0 & m_1-1 & 0 \\ m_1-1 & m_1\alpha & m_2(m_1-\alpha-1) \\ 0 & m_1(m_2-\alpha-1) & m_1(m_3-m_2+\alpha+1) \end{bmatrix}$$

$$p^3_{jk}=\begin{bmatrix} 0 & 0 & m_1-1 \\ 0 & m_1\beta & m_1(m_2-\beta) \\ m_1-1 & m_1(m_2-\beta) & m_1(m_3-m_2+\beta-1) \end{bmatrix}$$

Example. Consider the group G formed by $a^6=1$.

Let $A: a, a^3, a^5;\ B: a^2, a^4;$

So, $\alpha=0,\ \beta=3.$

With each element of G let us associate two treatments. Consider the first associate of θ_i as θ_j, when $j\neq i$, second associates of θ_i as $(\theta, \phi)_j$ where $\phi \in A$ and $j=1, 2$ and the third associates of θ_i as $(\theta.\psi)_j$ when $\psi \in B$ and $j=1, 2$. So, the parameters of the association scheme are

$$v=12,\ n_1=1,\ n_2=6,\ n_3=4.$$

$$p^1_{jk}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}\ p^2_{jk}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}\ p^3_{jk}=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 6 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Consider the set of six initial blocks $(1_1, a_1, 1_2)$, $(1_1, a^3_1, 1_2)$ $(1_1, a^5_1, 1_2)$, $(1_1, 1_2, a_2)$, $(1_1, 1_2, a^3_2)$, $(1_1, 1_2, a^5_2)$. On developing these blocks we get the solution of the design $v=12$, $b=36$, $r=9$, $k=3$, $\lambda_1=6$, $\lambda_2=2$, $\lambda_3=0$.

2.1.3. Consider a group $G=[a, c]$ where $a^{m_2+m_3+1}=c^n=1$. With each element of G, let us associate a set of t elements. Let $G_1\equiv 1, c, c^2, \ldots, c^{n-1}$; $G_2=1, a, a^2\ldots, a^{m_2+m_3}$. So, $G=G_1\otimes G_2$. Let further, the non-unit elements of G_2 be divided into two disjoint sets A and B of m_2 and m_3 elements respectively such that $A=A^{-1}$, and among the ratios arising out of A, the elements of A appear α times and elements of B appear β times.

Consider the first associates of any treatment θ_j as $[\theta\otimes(G_1-1)]_j$, $[\theta\otimes G_1]_k$, $k=1, 2, \ldots, j-1, j+1, \ldots t \ldots$, second associates of θ as $(\theta\oplus A\otimes G_1)_k$, $k=1, 2, \ldots, t$; the rest are third associates.

So, this will give us the three associate cyclic association scheme whose association parameters are $v=m_1(m_2+m_3+1)$, $n_1=m_1-1$, $n_2=m_1m_2$, $n_3=m_1m_3$ where $m_1=tn$

$$p^1_{jk}=\begin{bmatrix} m_1-2 & 0 & 0 \\ 0 & m_1m_2 & 0 \\ 0 & 0 & m_1m_3 \end{bmatrix}$$

$$p^2_{jk}=\begin{bmatrix} 0 & m_1-1 & 0 \\ m_1-1 & m_1\alpha & m_1(m_2-\alpha-1) \\ 0 & m_1(m_2-\alpha-1) & m_1(m_3-m_2+\alpha+1) \end{bmatrix}$$

$$p^3_{jk}=\begin{bmatrix} 0 & 0 & m^1-1 \\ 0 & m_1\beta & m_1(m_2-\beta) \\ m_1-1 & m_1(m_2-\beta) & m_1(m_3-m_2+\beta-1) \end{bmatrix}$$

Theorem 2.1.3. Let it be possible to select a set of t blocks $B_1, B_2, \ldots, B_t$ such that

(*i*) Each block contains k distinct treatments.

(*ii*) Among the kt treatments occurring in the t=blocks exactly r treatments should belong to each of m_1 distinct classes.

(*iii*) Among the ratios of the type (i, i), $i=1, 2, \ldots, m_1/n$ the non-unit elements of G_1 appear λ_1 times, and among the ratios of the type $\underset{i\neq j}{[i, j]}$ the elements of G_1 appear λ_1 times, the ratios of the type $[i, j]$ $i, j=1, 2, \ldots, m_1/n$ the elements while among of $G_1\otimes A$ appear λ_2 times and $G_1\otimes B$ appear λ_3 times.

Then on develping the blocks (i.e., multiplying successively by the elements of G) we get the solution of the design

$$v=m_1(m_2+m_3+1),\ b=(m_2+m_3+1)nt,\ r=k\ n\ t/m_1,\ k,\ \lambda_1,\ \lambda_2,\ \lambda_3.$$

Example. Consider the group G formed by a and c together with their different powers and product of their powers when $a^4=c^2=1$. With each element u of G let us associate two treatments u_1 and u_2. Let G_1 : 1, c; A : a, a^3 and B; a^2.

So, $\alpha=0$, $\beta=2$.

Consider the two initial blocks $[1_1, c_1, 1_2, c_2, (a^2)_1]$ and $[1_1, c_1, 1_2, c_2, (a^2)_2]$. The ratios arising out of these blocks satisfy the condition of theorem 2.1.3 with $\lambda_1=4$, $\lambda_2=0$, $\lambda_3=2$. Hence on developing these blocks (i.e. multiplying successively by the elements of the group G) we get the solution of the design

$$v=b=16,\ r=k=5,\ n_1=3,\ n_2=8,\ n_3=4,\ \lambda_1=4, \lambda_2=0,\ \lambda_3=2.$$

$$p^1_{jk}=\begin{bmatrix}2&0&0\\0&8&0\\0&0&4\end{bmatrix}\ p^2_{jk}=\begin{bmatrix}0&3&0\\3&0&4\\0&4&0\end{bmatrix}\ p^3_{jk}=\begin{bmatrix}0&0&3\\0&8&0\\3&0&0\end{bmatrix}$$

2.2. *Cyclic association scheme (second type)* :

2.2.1. Consider the abelian group G; the factor groups G_1, G_2 and the sets A and B of section 2.1. Let the first associates of any treatment θ be θA, its second associates be θB and its third associates $\theta G_2 \otimes (G_1-1)$. With each element of G let us associate a treatment.

The parameters of the association scheme are

$$v=m_1 n,\ n_1=m_2,\ n_2=m_3,\ n_3=(m_1-1)\ (m_2+m_3+1)$$

$$p^1_{jk}=\begin{bmatrix}\alpha & m_2-\alpha-1 & 0\\ m_2-\alpha-1 & m_3-m_2+\alpha+1 & 0\\ 0 & 0 & (m_1-1)n\end{bmatrix}$$

$$p^2_{jk}=\begin{bmatrix}\beta & m_2-\beta & 0\\ m_2-\beta & m_3-m_2+\beta-1 & 0\\ 0 & 0 & (m_1-1)n\end{bmatrix}$$

$$p^3_{jk}=\begin{bmatrix}0 & 0 & m_2\\ 0 & 0 & m_3\\ m_2 & m_3 & (m_1-2)n\end{bmatrix}$$

Theorem 2.2.1. Let it be possible to select a set of t-blocks $B_1, B_2 \ldots, B_t$ satisfying the following conditions :

(*i*) Every block contains k treatments. The treatments contained in the same block are all distinct.

(*ii*) Among the $tk(k-1)$ ratios arising out of these t-blocks the elements of A appear λ_1 times, the elements of B appear λ_2 times and the elements of $(G_1-1)\otimes G_2$ appear λ_3 times.

Hence on developing these blocks (*i.e.*, multiplying successively by the elements of G) we get the solution of the design whose parameters of the second kind are $v=m_1n$, $b=vt$, $r=kt$, k, λ_1, λ_2, λ_3.

Example 1. Consider the group G formed by a and c

when $a^6=c^2=1.$

Let $G_1=1, c; A: a, a^2, a^4, a^5; B: a^3.$

So, $\alpha=2,\ \beta=4.$

Hence the parameters of the association scheme are

$$v=12,\ n_1=4,\ n_2=1,\ n_3=6.$$

$$p^1_{jk}=\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad p^2_{jk}=\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad p^3_{jk}=\begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 1 \\ 4 & 1 & 0 \end{bmatrix}$$

Consider the initial block $(1, a^3, c, ac, a^4c)$. We can easily verify that this initial block satisfies the condition of theorem 2.4 having $\lambda_1=1$, $\lambda_2=4$, $\lambda_3=2$. Hence on developing this block we get the solution of the design whose parameters of the second kind are

$$v=b=12,\ r=k=5,\ \lambda_1=1,\ \lambda_2=4,\ \lambda_3=2.$$

2.2.2. Let $G=[a]$ be a α multiplicative abelian group of order m_2+m_3+1.

Let us divide the non-unit elements of G into two disjoint sets A and B with m_2 and m_3 elements respectively where $A^{-1}=A$ and among the ratios arising out of the elements of A, the elements of A appear α times and the elements of B appear β times.

To any element a^i let there correspond m_1 treatments $a^i_1, a^i_2, \ldots, a^i_{m_1}$. Treatments denoted by symbols with the same lower suffix j may be said to belong to the j-th class.

Consider the 1st associates of θ_i as $[\theta \otimes A]_i$, second associates of θ_i as $[\theta \oplus B]_i$; and rest as 3rd associates. Hence the parameters of the association scheme are

$$v=m_1 (m_2+m_3+1),\ n_1=m_2,\ n_2=m_3,\ n_3=(m_1-1)(m_2+m_3+1)$$

$$p^1_{jk}=\begin{bmatrix} \alpha & m_2-\alpha-1 & 0 \\ m_2-\alpha-1 & m_3-m_2+\alpha+1 & 0 \\ 0 & 0 & (m_1-1)(m_2+m_3+1 \end{bmatrix}$$

$$p^2_{jk}=\begin{bmatrix} \beta & m_2-\beta & 0 \\ m_2-\beta & m_3-m_2+\beta-1 & 0 \\ 0 & 0 & (m_1-1)(m_2+m_3+1) \end{bmatrix}$$

$$p^3_{jk}=\begin{bmatrix} 0 & 0 & m_2 \\ 0 & 0 & m_3 \\ m_2 & m_3 & (m_1-2)(m_2+m_3+1) \end{bmatrix}$$

Theorem 2.2.2. Let it be possible to select a set of t-blocks $B_1, B_2, \ldots, B_t$ such that

(*i*) Every block contains exactly k distinct treatments.

(*ii*) Among the kt treatments occurring in the t-blocks exactly r treatments should belong to each of m_1 distinct classes. Obviously, $m_1 r=kt$.

(*iii*) Among the ratios of the type $\lfloor i, i \rfloor$, $i=1, 2, \ldots, m_1$; arising out of these t-blocks, the elements of A appear λ_1 times and the elements of B appear λ_2 times; among the ratios of the type $\begin{bmatrix} i, j, \\ i \neq j \end{bmatrix}$ $i, j=1, 2, \ldots, m_1$; the elements of G appear λ_3 times.

Hence on developing the blocks (*i.e.*, multiplying successively by elements of the group G, we get the solution of the design $v=m_1 (m_2+m_3+1)$, $b=(m_2+m_3+1)\,t$, $r, k, \lambda_1, \lambda_2, \lambda_3$.

Example. Consider for example the group $G=[a]$

where $a^6=1$.

Let $A : a, a^3, a^5 ; B : a^2, a^4$

So, $\alpha=0, \beta=3$.

With each element of G let us associate two treatments. Consider the first associates of θ_i as $[\theta \oplus (a, a^3, a^5)]_i$; second associates of θ_i as $[\theta \oplus (a^2, a^4)]_i$ and the third associates of θ_i as $(1_j, a_j, a^2{}_j, a^3{}_j, a^4{}_j, a^5{}_j)$ where $j \neq i$. So the parameters of the association scheme are $v=12$, $n_1=3$, $n_2=2$, $n_3=6$.

$$p^1{}_{jk}=\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad p^2{}_{jk}=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad p^3{}_{jk}=\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

Consider the following two initial blocks

$$(1_1, a^2{}_1, a^4{}_1, 1_2) \text{ and } (1_2, a^2{}_2, a^4{}_2, a_1).$$

It is easy to verify that among the ratios of the type (1, 1) or (2, 2) the elements of A do not appear at all, while the elements of B appear thrice. But among the differences of the type (1, 2) or (2, 1) the elements of G appear once. So, on developing these blocks [*i.e.*, multiplying successively by $1, a, a^2, a^3, a^4, a^5$] we get the solution of the design

$$v=b=12, \quad r=k=4, \quad \lambda_1=0, \quad \lambda_2=3, \quad \lambda_3=1.$$

2.2.3. Consider a group G consisting of $n(m_2+m_3+1)$ elements, a, c and their different powers when $a^{m_2+m_3+1}=c^n=1$. With each element of G, let us associate a set of m_1/n treatments. (It is assumed that n is a divisor of m_1).

Let $G_1 \equiv 1, c, c^2, \ldots, c^{n-1}; G_2=1, a, a^2 \ldots a^{m_2+m_3}$.

So, $$G=G_1 \otimes G_2.$$

Let further $G_2 = A \cup B \cup \{1\}$, when $A=A^{-1}$, and among the ratios arising out of the elements of A, each element of A appears α times and each element of B appears β times.

Consider the 1st associates of any treatment θ_i as $(\theta \oplus A)_i$, second associates of θ_i as $(\theta \oplus B)_i$ and the rest are third associates.

The parameters of the association scheme are

$$n_1=m_2,\ n_2=m_3,\ n_3=(m_1-1)\ (m_2+m_3+1)$$

$$p^1_{jk}=\begin{bmatrix} \alpha & m_2-\alpha-1 & 0 \\ m_2-\alpha-1 & m_3-m_2+\alpha+1 & 0 \\ 0 & 0 & (m_1-1)\ (m_2+m_3+1) \end{bmatrix}$$

$$p^2_{jk}=\begin{bmatrix} \beta & m_2-\beta & 0 \\ m_2-\beta & m_3-m_2+\beta-1 & 0 \\ 0 & 0 & (m_1-1)\ (m_2+m_3+1) \end{bmatrix}$$

$$p^3_{jk}=\begin{bmatrix} 0 & 0 & m_2 \\ 0 & 0 & m_3 \\ m_2 & m_3 & (m-2)\ (m_1+m_3+1) \end{bmatrix}$$

Theorem 2.2.3. Let it be possible to select a set of t-blocks $B_1, B_2, \ldots, B_t$ such that

(*i*) Each block contains k treatments

(*ii*) Among the kt treatments occurring in the t-blocks exactly r treatments should belong to each of m_1 distinct classes.

(*iii*) Among the ratios of the type $[i, i]$ $i=1, 2, \ldots, m_1/n$ arising out of these t-blocks, the elements of A appear λ_1 times, the elements of B appear λ_2 times and the elements of $(G_1-1)\oplus G_2$ appear λ_3 times. While among the ratios of the type $[i, j]$ the elements of G appear λ_3 times ; $[i, j]$ $=1, 2, \ldots, m_j/n$. Then on developing these set of t blocks we shall get the solution of the design $v=m_1\ (m_2+m_3+1)$, $b=(m_2+m_3+1)$ t, r, k, λ_1, λ_2, λ_3, $n_1=m_2$, $n_2=m_3$, $n_3=(m_1-1)\ (m_2+m_3+1)$

$$p^1_{jk}=\begin{bmatrix} \alpha & m_2-\alpha-1 & 0 \\ m_2-\alpha-1 & m_3-m_2-\alpha-1 & 0 \\ 0 & 0 & (m_1-1)(m_2+m_3+1) \end{bmatrix}$$

$$p^2_{jk}=\begin{bmatrix} \beta & m_2-\beta & 0 \\ m_2-\beta & m_3-m_2+\beta-1 & 0 \\ 0 & 0 & (m_1-1)(m_2+m_3+1) \end{bmatrix}$$

$$p^3_{jk}=\begin{bmatrix} 0 & 0 & m_2 \\ 0 & 0 & m_3 \\ m_2 & m_3 & (m_1-2)(m_2+m_3+1) \end{bmatrix}$$

Example. Consider the group $G=[a, c]$ where $a^4=c^2=1$

Let $G_1 : 1, c\ ;\ A : a, a^3 : B : a^2$

So, $\alpha=0, \quad \beta=2.$

Consider the set of 12 initial blocks

$(1_1, (a^2)_1, c_1), (a_1, (a^3)_1, c_1), (1_2, (a^2)_2, c_2), (a_2, (a^3)_2, c_2),$

$(1_2, (a^2)_1, c_2), (a_1, (a^3)_1, c_2), (a_1, (a^3)_1, 1_2), (1_1, (a^2)_1, 1_2),$

$(1_1, 1_2, (a^2)_2), (1_2, (a^2)_2, c_1), ((a)_2, (a^3)_2, c_1), ((a^2)_3, (a^3)_1, 1_2)$

Then on developing these blocks (*i.e.*, multiplying successively by $1, a, a^2, a^3, c, ac, a^2c, a^3c$), we get the solution of the design $v=16$, $b=96$, $r=18$, $k=3$, $\lambda_1=0$, $\lambda_2=12$, $\lambda_3=2$, $n_1=2$, $n_2=1$, $n_3=12$.

$$p^1_{jk}=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 12 \end{bmatrix} \quad p^2_{jk}=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12 \end{bmatrix} \quad p^3_{jk}=\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 8 \end{bmatrix}$$

2.3. *Cyclic-triangular Association Scheme (First type)* :

Consider an abelian group $G=[a, c]$ where $a^{n(n-1)/2}=c^{ml}=1$.

Let $G=G_1 \oplus G_2$, where $G_1 : 1, c, c^2 \ldots, c^{m1-1}$.

Let the $n(n-1)/2$ elements of G_2 be divided into the following $(m-1)$ sets when $n=2m-1$:

$$S_0=(1, a^{m-1}, a^{2(m-1)}, \ldots, a^{(n-1)(m-1)})$$

$$S_1\equiv(a, a^m, a^{2\ m-1}, \ldots, a^{(n-1)\ (m-1)+1}) \qquad \ldots (2.3.1)$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$S_{m-2}\equiv(a^{m-2}, a^{2m-3}, a^{3m-4}, \ldots, a^{(n-1)\ (m-1)+(m-2)})$$

and when $n=2m$, be divided into the following $m-$sets :

$$S_0 \equiv (1,\ a^m\ a^{2m},\ \ldots,\ a^{(n-2)m})$$
$$S_1 \equiv (a,\ a^{m+1},\ a^{2m+1},\ \ldots,\ a^{(n-2)m+1})$$
$$\ldots \quad \ldots \quad \ldots \qquad \ldots (2.3.2.)$$
$$S_{m-1} \equiv (a^{m-1},\ a^{2m-1},\ a^{3m-1},\ a^{m(n-1)-1})$$

Let it be possible to select t sets (where $t=m-1$ if $n=2m-1$ and $t=m$ if $n=2m$ of elements A_0, A_1, ..., A_{t-1} from the non-unit elements of G_2, where

$$A_0 \equiv (a^{d_{01}},\ a^{d_{02}},\ \ldots,\ a^{d_{0n_1}})$$
$$A_1 \equiv (a^{d_{11}},\ a^{d_{12}},\ \ldots,\ a^{d_{1n1}})$$
$$\ldots (2.3.3)$$
$$A_{t-1} \equiv (a^{d_{t1}},\ a^{d_{t2}},\ \ldots,\ a^{d_{tn1}})$$

having the properties

(*i*) d_{ij}'s are all different for a fixed i.

(*ii*) If θ is any element of S_i and $\theta.\ a^{d_{ik}}$ belongs to S_j, then $a^{-d_{ik}}$ should belong to A_j.

(*iii*) If for any element θ belonging to S_i, $\theta.\ a^{d_{ik}}$ belongs to S_j then among the ordered ratios that arise by taking the ratios of all elements of A_j from every element of A_i (*i.e.*, all possible $a^{d_{il}}/a^{d_{jl}}$) the element $a^{d_{jk}}$ should be repeated $n-2$ times. If for any element θ belonging to S_i, $\theta.a^{e_{ik}}$, where $a^{e_{ik}}$ is any non-unit element of G_2-A_i, belong to S_j, then among the ordered ratios that arise by taking the ratios of all elements of A_j from every element of A_i, the element $a^{e_{ik}}$ should be repeated 4 times.

With each element of G let us associate a treatment. It follows from Adhikary (1969), that if we consider the 1st associates of any treatment $\theta \in S_i$ as $\theta \otimes (G_1-1)$ the second associates of θ as $\theta G_1 \otimes A_i$, third associates of θ as $\theta\ G_1 \otimes (G_2-A_i-1)$, the parameters of the association scheme will be $v=m_1 n(n-1)/2$,

$$n_1=m_1-1,\ n_2=2m_1(n-2),\ n_3=m_1\,(n-2)\,(n-3)/2$$

$$p^1_{jk}=\begin{bmatrix} m_1-1 & 0 & 0 \\ 0 & 2m_1(n-2) & 0 \\ 0 & 0 & m_1(n-2)\,(n-3)/2 \end{bmatrix}$$

$$p^2_{jk}=\begin{bmatrix} 0 & m_1-1 & 0 \\ m_1-1 & m_1(n-2) & m_1(n-3) \\ 0 & m_1(n-3) & m_1(n-3)\,(n-4)/2 \end{bmatrix}$$

$$p^3_{jk}=\begin{bmatrix} 0 & 0 & m_1-1 \\ 0 & 4m_1 & 2m_1\,(n-4) \\ m_1-1 & 2m_1\,(n-4) & m_1\,(n-4)\,(n-5)/2 \end{bmatrix}$$

[It may be noted that the first type of cyclic-triangular association scheme was obtained by Singh and Singh (1964). Adhikary (1966) obtained the same association scheme in a systematic manner independently of Singh and Singh. However, this representation of cyclic association scheme which is essential for cyclic generation of cycle-triangular P.B.I.B. design is due to Adhikary (1969).]

Theorem 2.3.1. Let it be possible to select a set of s blocks $B_1, B_2, \ldots, B_s$ satisfying the following conditions :

(*i*) Every block contains k treatments,

(*ii*) Among the ks treatments occurring in the s-blocks, the number of elements appearing from $S_j \otimes G_1$, $j=0, 1, 2, \ldots, t-1$ is constant, equal to r. Obviously, $ks=tr$.

(*iii*) The all possible ratios arising out of these s blocks can be classified in the following :

Consider the elements within blocks which belong to $S_j \oplus G_1$. Let α be any such element occurring in, say, the l-th block. Then form the ratios a_l/α where a_l is any element other than α in the l-th block. Among the ratios obtained from all the elements of S_j, occurring in the s blocks the elements of (G_1-1) will appear λ_1 times, the elements of $A_j \oplus G_1$ appear λ_2 times and other non-unit elements appear λ_3 times ; $j=0, 1, 2, \ldots, t-1$.

So,

$$t\left\{(m_1-1)\ \lambda_1+2m_1\ (n-2)\ \lambda_2+\frac{m_1}{2}(n-3)\ (n-4)\ \lambda_3\right\}=sk\ (k-1)$$

Then, on developing these blocks (*i.e.*, multiplying successively by 1, a^t. a^{2t}. . . . , $a^{(p-1)t}$, c, $a^t c$, . . , $a^{(p-1)t}c$, ...c^{m1-1}, $a^{(p-1)t}\ c^{m1-1}$,

where $$pt=\frac{n(n-1)}{2},$$

i.e., elements of $S_0 \otimes G_1$, we get the solution of the cyclic triangular design $v=m_1 n\ (n-1)/2$, $b=p\ s\ m_1$, r, k, λ_1, λ_2, λ_3.

Example : Consider the group G formed by $a^{10}=c^2=1$.

Let $G_1\equiv 1, c,\ G_2\equiv 1, a, a^2, \ldots, a^9$

$S_0\equiv(1, a^2, a^4, a^6, a^8)$ $A_0\equiv(a, a^2, a^3, a^7, a^8, a^9)$

$S_1\equiv(a, a^3, a^5, a^7, a^9)$ $A_1\equiv(a, a^3, a^4, a^6, a^7, a^9)$

So, $t=2$,

The parameters of the association scheme are

$v=20,\ n_1=1,\ n_2=12,\ n_3=6.$

$$p^1_{jk}=\begin{bmatrix}0&0&0\\0&12&0\\0&0&6\end{bmatrix}\quad p^2_{jk}=\begin{bmatrix}0&1&0\\1&6&4\\0&4&2\end{bmatrix}\quad p^3_{jk}=\begin{bmatrix}0&0&1\\0&8&4\\1&4&0\end{bmatrix}$$

Consider the set of six initial blocks

(a, a^3, ac), (a^2, a^8, a^8c), (a, a^3, a^3c), (a^2, a^8, a^2c), $(1, a^5, c)$

and $(1, a^5, a^5c)$.

We can easily verify that these blocks satisfies all the conditions of theorem 2.3.1 with $r=9$, $\lambda_1=6$, $\lambda_2=0$, $\lambda_3=2$.

Hence developing these blocks (*i.e.* multiplying these blocks successively by elements of $S_0 \oplus G_1$) we gat the solution of the design

$$v=20,\ b=60,\ r=9,\ k=3,\ \lambda_1=6,\ \lambda_2=0,\ \lambda_3=2.$$

2.4. *Cyclic-Triangular Association Scheme (Second Type) :*

Let us take the group G, G_1, G_2 ; the sets A_i's and S_is

$$(i=1, 2, \ldots t \text{ when, } t=m-1 \text{ if } n=2m-1 \text{ and } t=m \text{ if } n=2m)$$

as in article 2.3. With each element of G let us associete a treatment.

Consider the first associates of any treatment $\theta \in S_i$ as $\theta \oplus A_i$, second associates of θ as $\theta \oplus (G_2 - A_i - 1)$ and the third associates of θ as $\theta\ G_2 \oplus (G_1 - 1)$.

The parameters of the association scheme are

$$v=m_1 n(n-1)/2,\ n_1=2(n-2),\ n_2=(n-2)\ (n-3)/2,\ n_3=(m_1-1)n(n-1)/2.$$

$$p^1_{jk}=\begin{bmatrix} n-2 & n-3 & 0 \\ n-3 & \dfrac{(n+3)\ (n-4)}{2} & 0 \\ 0 & 0 & (m_1-1)n(n-1)/2 \end{bmatrix}$$

$$p^2_{jk}=\begin{bmatrix} 4 & 2(n-4) & 0 \\ 2(n-4) & (n-4)n(n-5)/2 & 0 \\ 0 & 0 & (m_1-1)n(n-1)/2 \end{bmatrix}$$

$$p^3_{jk}=\begin{bmatrix} 0 & 0 & 2(n-2) \\ 0 & 0 & (n-2)(n-3)/2 \\ 2(n-2) & (n-2)\ (n-3)/2 & (m_1-2)n(n-1)/2 \end{bmatrix}$$

Theorem 2.4.1. Let it be possible to select a set of s blocks B_1, B_2, ... B_s satisfying the following conditions.

(*i*) Every block contains k treatments

(*ii*) Among the ks treatments occurring in the s blocks the number of elements appearing from $S_j \oplus G_1$, $j=0, 1, \ldots, t-1$ is constant equal to r.

(*iii*) Among the different types of ratios that can be formed from the blocks by taking one element from $S_j \oplus G_1$, the elements of A_j appear λ_1, times, the elements of $(G_2 - A_j - 1)$ appear λ_2 times and the elements of $(G_1 - 1) \oplus G_2$ appear λ_3 times. Then on developing these t blocks (*i.e.*, multiplying successively by the elements of $S_0 \oplus G_1$), we get the solution of the design

$$v = m_1 n(n-1)/2, \; b = m_1 \; ps, \text{ when } pt = \frac{n(n-1)}{2} \; r, k, \lambda_1, \lambda_2, \lambda_3,$$

Example. Consider the group G formed by $a^{15} = c^3 = 1$.

Let $G_1 : 1, c, c^2$. $G_2 : 1, a, a^2, \ldots, a^{14}$. So, $t=3$.

$S_0 : (1, a^3, a^6, a^9, a^{12})$ $A_0 : (a, a^2, a^6, a^9, a^{11}, a^{12}, a^{13})$

$S_1 : (a, a^4, a^7, a^{10}, a^{13})$ $A_1 : (a, a^2, a^3, a^4, a^{10}, a^{12}, a^{13}, a^{14})$

$S_2 : (a^2, a^5, a^8, a^{11}, a^{14})$ $A_2 : (a^2, a^4, a^5, a^6, a^9, a^{11}, a^{13}, a^{14})$

So, the parameters of the association scheme are

$$v=45, n_1=8, n_2=6, n_3=30$$

Let us develop the initial blocks $(1, a^4, a^{10})$, $(1, a^5, a^8)$, $(1, a^7, a^{14})$ partially by multiplying them successively by elements of $S_0 \oplus G_0$ and develop the blocks $(1, a^2c, a^8c^2)$, $(1, a^9c, a^7c^2)$, $(1, a^4c, c^2)$, $(1, a^6c, a^5c^2)$ and $(1, a^7c, a^{10}c^2)$ completely by multiplying successively by the elements of the group G. Then we get the solution of the design,

$$v=45, b=270, \; r=18, \; k=3, \; \lambda_1=0, \lambda_2=1, \; \lambda_3=1.$$

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